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HIGHER DIMENSIONAL ANALOGUES OF THE MODULAR AND PICARD GROUPS

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ABSTRACT. Clifford algebras are used to describe arithmetic groups which are generalizations of the modular and Picard groups.

1. INTRODUCTION

The modular group $PSL(2, \mathbf{Z})$ and the Picard group $PSL(2, \mathbf{Z}(i))$ are distinguished as being simple examples of arithmetic groups with a rich geometric, algebraic and number theoretic structure. Since each is the orientation-preserving subgroup of a group generated by reflections in the faces of a Coxeter polyhedron in hyperbolic space they have presentations as a graph amalgamation product of finite vertex groups [4].

Ahlfors' description of hyperbolic isometries via Clifford algebras $\mathscr{C}_n$[1, 2] gives rise to a natural generalization of the above groups in higher dimensions. In §4 these are shown to be arithmetic and their relationship with $SO^+(n, 1, \mathbf{Z})$ is discussed. The only cases where the Clifford group coincides with the nonzero elements of the algebra are for $\mathscr{C}_0 = \mathbf{R}$, $\mathscr{C}_1 = \mathbf{C}$ and $\mathscr{C}_2 = \mathbf{H}$ (the quaternions). The corresponding groups $PSL(2, \mathbf{Z})$, $PSL(2, \mathbf{Z}(i))$ and $PSL(2, \mathbf{Z}(i, j))$ are described in §6 and a presentation of each is given as a graph amalgamation product of vertex groups.

In §5 we describe the set of points fixed by parabolic elements and those fixed by hyperbolic elements. Finally, in §7 we investigate the orbit of infinity.

2. DEFINITIONS AND NOTATION

Following the lead of Ahlfors [1, 2], let the *Clifford algebra* $\mathscr{C}_n$ be the associative algebra over the reals generated by $i_1, \dots, i_n$ and defined by the relations $i_m^2 = -1$ and $i_l i_m = -i_m i_l$ for $l \neq m$. As a real vector space $\mathscr{C}_n$ has dimension 2^n spanned by products $i_{v_1} \cdots i_{v_m}$ with $0 < v_1 < \cdots < v_m \leq n$ where the null product is identified with the real number 1. We let $\mathscr{C}_n(\mathbf{Z})$ denote the corresponding module of integer combinations of the above elements.

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A linear combination $x = x_0 + x_1 i_1 + \cdots + x_n i_n$ is called a *vector* and is identified with $(x_0, \ldots, x_n) \in \mathbf{R}^{n+1}$. Products of nonzero vectors form a group: the *Clifford group* Γ_n. General elements of $\mathscr{C}_n$ are called *multivectors.*

Three important involutions on $\mathscr{C}_n$ are:

(i) the *main involution,* $a \to a'$, obtained by replacing each i_m with $-i_m$, satisfying $(ab)' = a'b'$.

(ii) *reversion,* $a \to a^*$, obtained by reversing the order of the factors in each term $i_{v_1} \cdots i_{v_m}$, satisfying $(ab)^* = b^* a^*$.

(iii) *conjugation,* $a \to \overline{a} = (a')^* = (a^*)'$, satisfying $(\overline{ab}) = \overline{b}\overline{a}$.

Elements of Γ_n satisfy $|a|^2 = a\overline{a} = \overline{a}a$ and $|ab| = |a| \cdot |b|$.

Lemma 1 (Ahlfors [1]). *The following conditions on $a, b, c, d \in \Gamma_n$ with $ad^* - bc^* = 1$ are equivalent.*

(i) $ab^*,\ cd^*,\ c^*a,\ d^*b \in \mathbf{R}^{n+1}$,
(ii) $ab^*,\ cd^* \in \mathbf{R}^{n+1}$,
(iii) $ab^*,\ c^*a \in \mathbf{R}^{n+1}$,
(iv) $cd^*,\ c^*a \in \mathbf{R}^{n+1}$,
(v) $ab^*,\ d^*b \in \mathbf{R}^{n+1}$,
(vi) $cd^*,\ d^*b \in \mathbf{R}^{n+1}$.

Proof. Note that

$$c^*a = c^*(d'\overline{a} - c'\overline{b})a = c^*d'|a|^2 - |c|^2\overline{b}a,$$
$$d^*b = d^*(d'\overline{a} - c'\overline{b})b = |d|^2\overline{a}b - d^*c'|b|^2.$$

For $x, y \in \Gamma_n$ the following are seen to be equivalent:

(i) $xy^* \in \mathbf{R}^{n+1}$, (iii) $x^*y' \in \mathbf{R}^{n+1}$,
(ii) $yx^* \in \mathbf{R}^{n+1}$, (iv) $\overline{y}x \in \mathbf{R}^{n+1}$.

The result follows easily.

We now define the *ring of Clifford matrices*

$$M(2, \mathscr{C}_n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathscr{C}_n \right\},$$

the *group of Clifford matrices*

$$SL(2, \mathscr{C}_n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \Gamma_n \cup \{0\},\ ab^* \text{ and } cd^* \in \mathbf{R}^{n+1}, ad^* - bc^* = 1 \right\}$$

and the *group of projective Clifford matrices*

$$PSL(2, \mathscr{C}_n) = SL(2, \mathscr{C}_n)/\{\pm 1\}.$$

Thus

$$SL(2,\mathscr{C}_0(\mathbf{Z})) = SL(2,\mathbf{Z}), \quad SL(2,\mathscr{C}_1(\mathbf{Z})) = SL(2,\mathbf{Z}(i)),$$
$$SL(2,\mathscr{C}_2(\mathbf{Z})) = SL(2,\mathbf{Z}(i,j)).$$

Theorem 1 (Ahlfors [2]). *$SL(2,\mathscr{C}_n)$ acts on $\widehat{\mathbf{R}}^{n+2} = \mathbf{R}^{n+2} \cup \{\infty\}$ by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} x = (ax+b)(cx+d)^{-1}$$

preserving the upper half space H^{n+2} with its Poincaré metric. $PSL(2,\mathscr{C}_n)$ is isomorphic to the full group of orientation preserving hyperbolic isometries.

Hence $PSL(2,\mathscr{C}_n)$ is isomorphic to $SO^+(n+2,1)$, the group of future-preserving Lorentz transformations in dimension $n+2$ with positive determinant.

3. The groups $PSL(2,\mathscr{C}_n)$ and $SO^+(n+2,1)$

A specific isometry from the Poincaré model H^{n+2} to the Lorentz model of hyperbolic $n+2$-space induces an isomorphism between $PSL(2,\mathscr{C}_n)$ and $SO^+(n+2,1)$. We will write $\widetilde{A}$ for the image in $SO^+(n+2,1)$ of an element A of $PSL(2,\mathscr{C}_n)$. Furthermore, we write

$$\widetilde{A} = \left(\begin{array}{c|cc} \widehat{A} & \gamma & \delta \\ \hline \alpha & p & q \\ \beta & r & s \end{array}\right)$$

where $\widehat{A}$ is an $(n+1)\times(n+1)$ real matrix, α, β, γ, $\delta \in \mathbf{R}^{n+1}$ and $p,q,r,s \in \mathbf{R}$. Then

$$\widetilde{A}^{-1} = \left(\begin{array}{c|cc} \widehat{A}^t & \alpha & -\beta \\ \hline \gamma & p & -r \\ \delta & -q & s \end{array}\right)$$

and $\widetilde{A} \in SO^+(n+2,1)$ implies that $s > 0$.

Let

$$\lambda = -p-q+r+s, \quad \nu = -p+q-r+s,$$
$$\mu_A = p-q-r+s, \quad \mu_{A^{-1}} = p+q+r+s.$$

In order to describe the action of $SO^+(n+2,1)$ on H^{n+2}, we extend the $(n+1)\times(n+1)$ matrix $\widehat{A}$ to the $(n+2)\times(n+2)$ matrix $\left(\begin{smallmatrix} \widehat{A} & 0 \\ 0 & 1 \end{smallmatrix}\right)$. This corresponds to the Poincaré extension from $\widehat{\mathbf{R}}^{n+1}$ to H^{n+2}.

Lemma 2 (Greenberg [8]). *If $A \in PSL(2,\mathscr{C}_n)$ with $\widetilde{A}$ as above and $x \in H^{n+2}$ then*

$$Ax = \frac{(\delta+\gamma)|x|^2 + 2\widehat{A}x + (\delta-\gamma)}{\lambda|x|^2 + 2(\beta-\alpha)\cdot x + \mu_A}.$$

Lemma 3.

$$Ax = \frac{\lambda[(\delta+\gamma)|x|^2 + 2\widehat{A}x + (\delta-\gamma)]}{|\lambda x + (\beta-\alpha)|^2} \qquad (\lambda \neq 0).$$

Proof. Since elements of $SO^+(n+2, 1)$ preserve the quadratic form $x_1^2 + \cdots + x_{n+2}^2 - x_{n+3}^2$, the inner product of two distinct columns of $\widetilde{A}$ is 0. The inner product of a column or row with itself is $+1$ for the first $n+2$ columns or rows and -1 for the last column or row.

The result follows on noting that

$$\begin{aligned}|\beta-\alpha|^2 &= |\alpha|^2 + |\beta|^2 - 2\alpha\cdot\beta \\ &= (1-p^2+q^2) + (-1-r^2+s^2) - 2(qs-pr) \\ &= (-q+s)^2 - (-p+r)^2 = \lambda\mu_A.\end{aligned}$$

Lemma 4.

$$|\beta-\alpha|^2 = \lambda\mu_A, \quad |\delta-\gamma|^2 = \nu\mu_A,$$
$$|\beta+\alpha|^2 = \nu\mu_{A^{-1}}, \quad |\delta+\gamma|^2 = \lambda\mu_{A^{-1}}.$$

Proof. The computations are similar to those in Lemma 3.

Lemma 5. *If* $A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in PSL(2, \mathscr{C}_n)$ *then*

$$Ax = \frac{2|c|^2[2a\overline{c}|x|^2 + 2(ax\overline{d} + b\overline{x}\,\overline{c}) + 2b\overline{d}]}{|2|c|^2x + 2\overline{c}d|^2}.$$

Proof.

$$\begin{aligned}Ax = (ax+b)(cx+d)^{-1} &= \frac{(ax+b)(\overline{x}\,\overline{c}+\overline{d})}{|cx+d|^2} \\ &= \frac{a\overline{c}|x|^2 + ax\overline{d} + b\overline{xc} + b\overline{d}}{|cx+d|^2} \\ &= \frac{2|c|^2[2a\overline{c}|x|^2 + 2(ax\overline{d} + b\overline{x}\,\overline{c}) + 2b\overline{d}]}{|2\overline{c}|^2|cx+d|^2} \\ &= \frac{2|c|^2[2a\overline{c}|x|^2 + 2(ax\overline{d} + b\overline{x}\,\overline{c}) + 2b\overline{d}]}{|2|c|^2x + 2\overline{c}d|^2}.\end{aligned}$$

Lemma 6.

$$\frac{1}{|c|^2} = |A^{-1}\infty|\cdot|A0 - A\infty| \qquad (c \neq 0)$$

Proof.

$$\begin{aligned}|A^{-1}\infty|\cdot|A0 - A\infty| &= |-c^{-1}d|\cdot|bd^{-1} - ac^{-1}| \\ &= \frac{1}{|c|^2}\cdot|bc^* - a(c^{-1}d)c^*| \\ &= \frac{1}{|c|^2} \quad \text{since } ad^* - bc^* = 1 \text{ and } c^{-1}d \in \mathbf{R}^{n+1}.\end{aligned}$$

Lemma 7. *With A and $\tilde{A}$ as above, we have*

$$\begin{aligned} &\text{(i)}\quad \lambda = 2|c|^2, &&\text{(ii)}\quad 2p = |a|^2 - |b|^2 - |c|^2 + |d|^2, \\ &\qquad \mu_A = 2|d|^2, &&\qquad 2q = |a|^2 + |b|^2 - |c|^2 - |d|^2, \\ &\qquad \mu_{A^{-1}} = 2|a|^2, &&\qquad 2r = |a|^2 - |b|^2 + |c|^2 - |d|^2, \\ &\qquad \nu = 2|b|^2, &&\qquad 2s = |a|^2 + |b|^2 + |c|^2 + |d|^2; \end{aligned}$$

$$\begin{aligned} \text{(iii)}\quad \alpha &= \bar{a}b - \bar{c}d, \\ \beta &= \bar{a}b + \bar{c}d, \\ \gamma &= a\bar{c} - b\bar{d}, \\ \delta &= a\bar{c} + b\bar{d}. \end{aligned}$$

Proof. By Lemma 2 $A0 = (\delta - \gamma)/\mu_A$ and $A\infty = (\delta + \gamma)/\lambda$ and by Lemma 3 $A^{-1}\infty = (\alpha - \beta)/\lambda$. Thus by Lemma 6

$$\begin{aligned} \frac{|\lambda|^2}{4|c|^4} &= \frac{|\alpha - \beta|^2}{4}\left|\frac{\delta - \gamma}{\mu_A} - \frac{\delta + \gamma}{\lambda}\right|^2 \\ &= \frac{1}{4\lambda\mu_A}|\lambda(\delta - \gamma) - \mu_A(\delta + \gamma)|^2 \quad \text{by Lemma 4} \\ &= \frac{1}{4\lambda\mu_A}[\lambda^2|\delta - \gamma|^2 + \mu_A^2|\delta + \gamma|^2 - 2\lambda\mu_A(\delta - \gamma)\cdot(\delta + \gamma)] \\ &= \frac{1}{4}[\lambda\nu + \mu_A\mu_{A^{-1}} - 2(|\delta|^2 - |\gamma|^2)] \quad \text{by Lemma 4} \\ &= \frac{1}{4}[(-p + s)^2 - (q - r)^2 + (p + s)^2 - (q + r)^2 \\ &\qquad\qquad - 2(s^2 - q^2 - 1) + 2(r^2 - p^2 + 1)] \\ &= 1. \end{aligned}$$

Thus $\lambda = \pm 2|c|^2$. By comparing Lemma 3 and Lemma 5, one obtains

$$\begin{aligned} |2\bar{c}d|^2 &= |\beta - \alpha|^2 = \lambda\mu_A \quad \text{so} \quad \mu_A = \pm 2|d|^2, \\ |2a\bar{c}|^2 &= |\delta + \gamma|^2 = \lambda\mu_{A^{-1}} \quad \text{so} \quad \mu_{A^{-1}} = \pm 2|a|^2, \\ |2b\bar{d}|^2 &= |\delta - \gamma|^2 = \nu\mu_A \quad \text{so} \quad \nu = \pm 2|b|^2. \end{aligned}$$

Since $4s = \lambda + \nu + \mu_A + \mu_{A^{-1}}$ and $s > 0$, this proves (i) and therefore (ii). Further comparison of Lemma 3 and Lemma 5 together with the corresponding expressions for $A^{-1}x$ gives (iii).

4. The groups $PSL(2, \mathscr{C}_n(\mathbf{Z}))$ and $SO^+(n+2, 1, \mathbf{Z})$

Let $M(n, \mathbf{C})$ denote the ring of complex $n \times n$ matrices and $\mathrm{GL}(n, \mathbf{C})$ the subgroup of invertible matrices. For a ring $A \subset \mathbf{C}$ define

$$\mathrm{GL}(n, A) = \{(a_{ij}) \in \mathrm{GL}(n, \mathbf{C}) \colon a_{ij} \in A \text{ and } \det(a_{ij})^{-1} \in A\}.$$

By an *algebraic group defined over Q* we mean a subgroup G of $\mathrm{GL}(n, \mathbf{C})$ consisting of all those invertible matrices whose coefficients annihilate some finite collection of polynomials in n^2 variables with rational coefficients. For an algebraic group G we define

$$G_A = G \cap \mathrm{GL}(n, A)$$

and term $G_{\mathbf{Z}}$ an *arithmetic subgroup* of $G_{\mathbf{R}}$. Note that $G_{\mathbf{Z}}$ is obviously discrete.

Further, let G be a connected semisimple Lie group with finite center and no compact factors. A subgroup Γ of G is *arithmetic* if there exist (i) a semisimple algebraic group H defined over Q and (ii) an epimorphism $\psi: H_{\mathbf{R}}^0 \to G$ with compact kernel such that $\psi(H_{\mathbf{Z}} \cap H_{\mathbf{R}}^0)$ and Γ are commensurable. Here $H_{\mathbf{R}}^0$ is the component of the identity in $H_{\mathbf{R}}$. We recall that Γ, Γ' are *commensurable* if $[\Gamma: \Gamma \cap \Gamma']$ and $[\Gamma': \Gamma \cap \Gamma']$ are finite. See [3, 9, 17] for more details.

Thus $G_{\mathbf{Z}} = SO(n, 1, \mathbf{Z})$ is an arithmetic subgroup of $G_{\mathbf{R}} = SO(n, 1)$. Hence $SO^+(n+2, 1, \mathbf{Z})$ is also arithmetic. Further, it is of finite covolume and is finitely generated by virtue of the next theorem.

Theorem 2 (Borel and Harish-Chandra [3]). *If G is a semisimple algebraic Lie group, defined over Q, then $G_{\mathbf{Z}}$ is finitely generated and $G_{\mathbf{R}}/G_{\mathbf{Z}}$ has finite Haar measure.*

Theorem 3. *$SL(2, \mathscr{C}_n(\mathbf{Z}))$ is a finitely generated arithmetic group of finite covolume.*

Proof. For $a \in \mathscr{C}_n$ consider the map $\rho_a: \mathscr{C}_n \to \mathscr{C}_n$ given by $\rho_a(x) = ax$. Ordering the standard basis $1, i_1, \ldots, i_n, i_1 i_2, \ldots, i_1 i_2 \cdots i_n$ for $\mathscr{C}_n$ as a vector space over $\mathbf{R}$, we may view ρ_a as a linear map from $\mathbf{R}^{2^n}$ to itself with matrix A, say.

It is shown in Porteous [13] that the elements of Γ_n consist precisely of those invertible elements $a \in \mathscr{C}_n$ such that if $x \in \mathbf{R}^{n+1}$, then $axa'^{-1} \in \mathbf{R}^{n+1}$. Equivalently, $\Gamma_n \cup \{0\}$ consists of those $a \in \mathscr{C}_n$ such that $a\bar{a} = \bar{a}a = |a|^2$ and $ai_k a^*$ belongs to $\mathbf{R}^{n+1}$ for all k. These conditions are expressed by the coefficients of a satisfying finitely many polynomial equations. Thus the image of Γ_n inside $M(2^n, \mathbf{R})$ is the set of real points of an algebraic subgroup of $\mathrm{GL}(2^n, \mathbf{C})$ defined over Q.

Observe that if a is one of the basis elements, then A is, up to the signs of its entries, a permutation matrix. In general, the coefficients of a in their standard order form the first column of A. The other columns of A are permutations of the coefficients of a, up to signs. It follows that $a \in \mathscr{C}_n(\mathbf{Z})$ iff $A \in M(2^n, \mathbf{Z})$.

Now consider $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M(2, \mathscr{C}_n)$ and the linear map of $\mathscr{C}_n \times \mathscr{C}_n \to \mathscr{C}_n \times \mathscr{C}_n$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

We may view the above as a linear map from $\mathbf{R}^{2^{n+1}}$ with matrix T^*, say. Now T^* can be thought of as consisting of four 2^n by 2^n blocks determined as in the first part of this proof. Thus the map $T \to T^*$ gives an isomorphism from $SL(2, \mathscr{C}_n)$ onto the real points $G_{\mathbf{R}}$ of an algebraic subgroup of $\mathrm{GL}(2^{n+1}, \mathbf{C})$ defined over Q. Under this map $SL(2, \mathscr{C}_n(\mathbf{Z}))$ is identified with the set $G_{\mathbf{Z}}$ of integer points, which is a finitely generated arithmetic group by Theorem 2. This completes the proof of Theorem 3.

Theorem 4. *If* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL(2, \mathscr{C}_n(\mathbf{Z}))$ *and* $|a|^2 + |b|^2 + |c|^2 + |d|^2$ *is an even integer then* $\widetilde{A} \in SO^+(n+2, 1, \mathbf{Z})$. *In particular, if*

$$G_n(2) = \{A \in PSL(2, \mathscr{C}_n(\mathbf{Z})) \colon A \equiv I \ (\mathrm{mod}\, 2\},$$

then

$$\widetilde{G}_n(2) \text{ is a subgroup of } SO^+(n+2, 1, \mathbf{Z}).$$

Proof. The result is an immediate consequence of Lemmas 3, 5, and 7.

By the proof of Theorem 3, the image of $G_n(2)$ in $\mathrm{GL}(2^{n+1}, \mathbf{C})$ contains the principal congruence subgroup of level two in $G_{\mathbf{Z}}$ and so $G_n(2)$ is of finite index in $PSL(2, \mathscr{C}_n(\mathbf{Z}))$.

Corollary. $SO^+(n+2, 1, \mathbf{Z})$ *and* $P\widetilde{S}L(2, \mathscr{C}_n(\mathbf{Z}))$ *are commensurable arithmetic groups.*

Proof. The subgroup $\widetilde{G}_n(2)$ is of finite index in $P\widetilde{S}L(2, \mathscr{C}_n(\mathbf{Z}))$ and hence in $SO^+(n+2, 1, \mathbf{Z})$ since both $SO^+(n+2, 1, \mathbf{Z})$ and $PSL(2, \mathscr{C}_n(\mathbf{Z}))$ have finite covolume by Theorems 2, 3. This proves the corollary.

We now give some examples of T and $\widetilde{T}$ which show that neither $P\widetilde{S}L(2, \mathscr{C}_n(\mathbf{Z})) < SO^+(n+2, 1, \mathbf{Z})$ nor $SO^+(n+2, 1, \mathbf{Z}) < P\widetilde{S}L(2, \mathscr{C}_n(\mathbf{Z}))$ $(n > 0)$.

(i) $$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \widetilde{T} = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{I} & -1 & 1 \\ & & 0 & 0 \\ \hline 1 & 0 & 1/2 & 1/2 \\ 1 & 0 & -1/2 & 3/2 \end{array}\right),$$

so $T \in PSL(2, \mathscr{C}_n(\mathbf{Z}))$ but $\widetilde{T} \notin SO^+(n+2, 1, \mathbf{Z})$.

(ii) $$T = \begin{pmatrix} \frac{1+i}{\sqrt{2}} & 0 \\ 0 & \frac{1-i}{\sqrt{2}} \end{pmatrix}, \qquad \widetilde{T} = \left(\begin{array}{cc|c} 0 & -1 & \\ 1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & I \end{array}\right).$$

So $T \notin PSL(2, \mathscr{C}_n(\mathbf{Z}))$ but $\widetilde{T} \in SO^+(n+2, 1, \mathbf{Z})$ $(n > 0)$. Observe also

(iii) $$T = \begin{pmatrix} 1 & 1+i \\ 0 & 1 \end{pmatrix}, \qquad \widetilde{T} = \left(\begin{array}{cc|cc} & & -1 & 1 \\ \multicolumn{2}{c|}{I} & -1 & 1 \\ & & \multicolumn{2}{c}{0} \\ \hline 1 & 1 & 0 & 1 \\ 1 & 1 \ \ 0 & -1 & 2 \end{array}\right).$$

Here, $T \in PSL(2, \mathscr{C}_n)(\mathbf{Z}))$ and $\widetilde{T} \in SO^+(n+2, 1, \mathbf{Z})$ but $T \notin G_n(2)$.

5. Fixed points of elements of $PSL(2, \mathscr{C}_n(\mathbf{Z}))$

Since $PSL(2, \mathscr{C}_n(\mathbf{Z}))$ acts with finite volume quotient on H^{n+2}, its limit set in $\widehat{\mathbf{R}}^{n+2}$ is $\widehat{\mathbf{R}}^{n+1}$. For the modular group $PSL(2, \mathbf{Z})$ each rational number is a fixed point of a parabolic element and each quadratic $\alpha+\beta\sqrt{m}$ with $\alpha, \beta \in Q$ is a fixed point of a hyperbolic element. See Nicholls [12].

We show that this holds in general.

Recall that $T \in PSL(2, \mathscr{C}_n)$ is *hyperbolic* if it is conjugate to $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ with $\lambda \in \mathbf{R} - \{\pm 1\}$ and *strictly parabolic* if it is conjugate to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

We define the *trace* τ of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ by $\tau = a + d^*$.

Theorem 5. *If $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL(2, \mathscr{C}_n)$ then*

(i) *[Ahlfors [1]] T is hyperbolic iff $\tau \in \mathbf{R}$, $\tau^2 > 4$ and $c = c^*$.*
(ii) *T is strictly parabolic iff $\tau = \pm 2$ and $c = c^*$.*

In either case $c \in \mathbf{R}^{n+1}$, and such τ are conjugacy invariant.

Proof. If $c = 0$ the result is elementary, so assume $c \neq 0$ and that T fixed $\eta \neq \infty$. Then, if $S = \left(\begin{smallmatrix} 1 & -\eta-1 \\ 1 & -\eta \end{smallmatrix}\right)$

$$STS^{-1} = \begin{pmatrix} -a\eta - b + (\eta+1)(c\eta+d) & * \\ 0 & b - \eta d + (a - \eta c)(\eta + 1) \end{pmatrix}$$

$$= \begin{pmatrix} c\eta + d & * \\ 0 & a - \eta c \end{pmatrix} \quad \text{since } T\eta = T^{-1}\eta = \eta.$$

Thus if T is hyperbolic then $c\eta + d = \lambda \in \mathbf{R} - \{\pm 1\}$ and hence $c = \lambda(\eta + c^{-1}d)^{-1} \in \mathbf{R}^{n+1}$ and $\tau = a + d^* = (c\eta + d) + (a - \eta c)^* = \lambda + 1/\lambda$. Likewise if T is strictly parabolic $c = \pm(\eta + c^{-1}d)^{-1} \in \mathbf{R}^{n+1}$ and $\tau = \pm 2$.

Conversely, if $\tau \in \mathbf{R}$, $\tau^2 \geq 4$ and $c = c^*$ then

$$STS^{-1} = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{*-1} \end{pmatrix}$$

with $\tau = \alpha + \alpha^{-1}$. Since $\tau^2 \geq 4$, $\alpha \in \mathbf{R}$ and the theorem follows.

Lemma 8 (Ahlfors [1]). *If $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is hyperbolic or strictly parabolic, then T fixes $\frac{1}{2}(a - d^* \pm \sqrt{\tau^2 - 4})c^{-1}$.*

Proof. The condition $T\eta = \eta$ is equivalent to each of the following equations.

$$a\eta + b = \eta(c\eta + d),$$
$$a(\eta c) + bc^* = (\eta c)^2 + (\eta c)d^* \quad \text{since } c = c^* \text{ by Theorem 5},$$
$$(\eta c)^2 + (\eta c)d^* - a(\eta c) - ad^* = -1 \quad \text{since } ad^* - bc^* = 1,$$
$$(\eta c - a)(\eta c + d^*) = -1.$$

The given point(s) satisfy the last equation.

Theorem 6. *Each point of $\widehat{Q}^{n+1}$ is a fixed point of a strictly parabolic element of $PSL(2, \mathscr{C}_n(\mathbf{Z}))$. Conversely, every parabolic fixed point is an element of $\widehat{Q}^{n+1}$.*

Proof. Given $\mu \in \mathbf{Z}^{n+1}$ and $m \in \mathbf{Z}$

$$T = \begin{pmatrix} 1 + m|\mu|^2 & -|\mu|^2\mu \\ m^2\overline{\mu} & 1 - m|\mu|^2 \end{pmatrix}$$

fixes μ/m. T is strictly parabolic by Theorem 5.

The converse follows from Lemma 8 on noting that if $T \in PSL(2, \mathscr{C}_n(\mathbf{Z}))$ is parabolic then some power is strictly parabolic.

Theorem 7. *Given* $m \in \mathbf{Z}^+$, *not a square, and* $\alpha, \beta \in Q^{n+1}$ *with* $\beta \neq 0$, *there exists a hyperbolic element of* $PSL(2, \mathscr{C}_n(\mathbf{Z}))$ *fixing* $\alpha \pm \beta\sqrt{m}$. *Conversely, all hyperbolic fixed points are of this form.*

Proof. Choose $c_0 \in \mathbf{Z}^{n+1}$ such that $\alpha_0 = \alpha c_0 \in \mathbf{Z}^{n+1}$ and $\beta_0 = \beta c_0 \in \mathbf{Z}^+$. Let $u = c_1 \in \mathbf{Z}^+$ and $v = b_1 \in \Gamma_n(\mathbf{Z})$ be a solution of $u(\beta_0^2 m - \alpha_0^2) = v c_0$ and consider Pell's equation

$$x^2 - (\beta_0^2 c_1^2 m) y^2 = 1.$$

Let $x = \tau/2$, $y = \eta$ be a solution and put $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where

$$c = c_0 c_1 \eta, \quad a = \tau/2 + \alpha c,$$
$$b = b_1 \eta, \quad d = \tau/2 - \alpha c.$$

Then

$$a + d^* = \tau,$$
$$c^* a = c^*(\tau/2 + \alpha c) = (\tau/2)c^* + c^* \alpha c \in \mathbf{R}^{n+1},$$
$$c d^* = c(\tau/2 - \alpha c^*) = (\tau/2)c - c\alpha c^* \in \mathbf{R}^{n+1};$$

$$\begin{aligned} ad^* - bc^* &= (\tau/2 + \alpha c)(\tau/2 - \alpha c) - bc^* \\ &= \tau^2/4 - (\alpha c)^2 - bc^* \\ &= 1 + (\beta_0^2 c_1^2 m)\eta^2 - c_1(\beta_0^2 m - \alpha_0^2) c_1 \eta^2 - (\alpha c_0 c_1 \eta)^2 \\ &= 1. \end{aligned}$$

So $T \in PSL(2, \mathscr{C}_n(\mathbf{Z}))$. Further, since τ is an even integer and $c \in \mathbf{R}^{n+1}$, it follows from Theorem 5 that T is hyperbolic. To see that T does indeed fix $\alpha \pm \beta\sqrt{m}$ note that $\frac{1}{2}(a - d^*)c^{-1} = \alpha$ and that $(\tau/2)^2 - (\beta_0 c_1 \eta)^2 m = 1$. Thus $(\tau/2)^2 - (\beta c)^2 m = 1$ and hence $(\frac{1}{2}\sqrt{\tau^2 - 4})c^{-1} = \pm\beta\sqrt{m}$. The theorem now follows by Lemma 8.

6. A PRESENTATION FOR $PSL(2, \mathbf{Z}(i, j))$

The modular group $PSL(2, \mathbf{Z})$ is the orientation-preserving subgroup of the group generated by reflections in the sides of a hyperbolic triangle with angles $\pi/2$, $\pi/3$, 0. It is thus the free product of cyclic groups generated by

$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

which stabilize the finite vertices [11]. As such it may be viewed as a (trivial) graph amalgamation product of the vertex groups $\langle a : a^2 = 1 \rangle$, $\langle d : d^3 = 1 \rangle$.

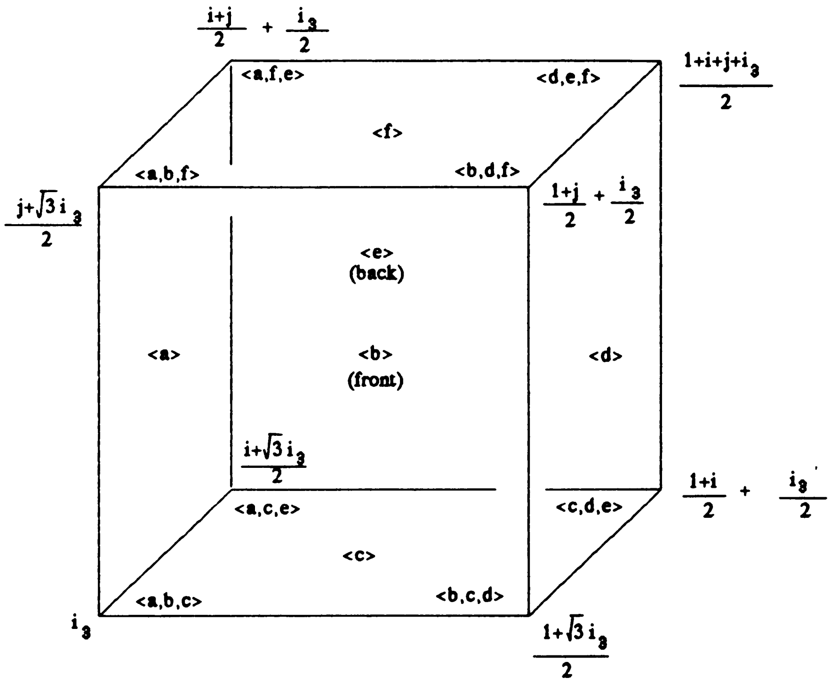


FIGURE 1

The Picard group $PSL(2, \mathbf{Z}(i))$ has a presentation as a graph amalgamation product of finite groups. This is a consequence of it being the orientation-preserving subgroup of the group generated by reflections in the faces of a Coxeter polyhedron in H^3. The polyhedron for $PSL(2, \mathbf{Z}(i))$ is the cone from ∞ to the quadrilateral lying on $H_2 = \{(x, y, t): x^2 + y^2 + t^2 = 1, t > 0\}$ and projecting onto the square in $\mathbf{R}^2$ with vertices at 0, $1/2$, $i/2$, and $(1+i)/2$. So the vertices of the quadrilateral are j, $(1+\sqrt{3}j)/2$, $(i+\sqrt{3}j)/2$ and $(1+i)/2 + j/\sqrt{2}$. The stabilizers of the four finite vertices are spherical triangle groups and the Picard group is presented as the graph amalgamated product of the vertex groups [4].

$$\langle a, b: a^2 = b^2 = (ab)^2 = 1\rangle: \text{the Klein 4-group } \mathbf{Z}_2 \oplus \mathbf{Z}_2,$$
$$\langle b, d: b^2 = d^3 = (bd)^2 = 1\rangle: \text{the dihedral group } D_6,$$
$$\langle d, e: d^3 = e^3 = (de^{-1})^2 = 1\rangle: \text{the tetrahedral group } A_4,$$
$$\langle e, a: e^3 = a^2 = (ea)^2 = 1\rangle: \text{the dihedral group } D_6,$$

where a and d are as above and

$$b = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad e = \begin{pmatrix} -1 & i \\ i & 0 \end{pmatrix}.$$

Similarly, the polyhedron in H^4 associated with $PSL(2, \mathbf{Z}(i, j))$ is the cone from ∞ to the "cube" lying on $H_3 = \{x_0 + x_1 i + x_2 j + x_3 i_3: \sum x_i^2 = 1$ and

Table 1

1. 

order $= 2mn$

$\langle a, b, c : a^n = b^2 = c^2 = (ab)^2 = (bc)^m = (ca)^2 = 1\rangle$

2. 

order $= 24$

$\langle a, b, c : a^2 = b^2 = c^2 = (ab)^3 = (bc)^3 = (ca)^2 = 1\rangle$

3. 

order $= 48$

$\langle a, b, c : a^2 = b^2 = c^2 = (ab)^3 = (bc)^4 = (ca)^2 = 1\rangle$

4. 

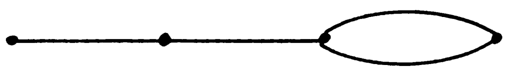
order $= 120$

$\langle a, b, c : a^2 = b^2 = c^2 = (ab)^3 = (bc)^5 = (ca)^2 = 1\rangle$

5. 

order $= 60$

$\langle a, b, c : a^3 = b^2 = c^2 = (ab)^3 = (bc)^3 = (ca)^2 = 1\rangle$

6. 

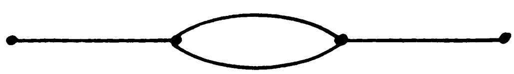
order $= 192$

$\langle a, b, c : a^3 = b^2 = c^2 = (ab)^3 = (bc)^4 = (ca)^2 = 1\rangle$

7. 

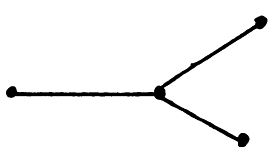
order $= 7200$

$\langle a, b, c : a^3 = b^2 = c^2 = (ab)^3 = (bc)^5 = (ca)^2 = 1\rangle$

8. 

order $= 576$

$\langle a, b, c : a^3 = b^2 = c^2 = (ab)^4 = (bc)^3 = (ca)^2 = 1\rangle$

9. 

order $= 96$

$\langle a, b, c : a^3 = b^2 = c^2 = (ab)^3 = (bc)^2 = (ca)^3 = 1\rangle$

$x_3 > 0\}$ and projecting onto the cube in $\mathbf{R}^3$ with vertices at

$$0, 1/2, i/2, (1+i)/2, j/2, (1+j)/2, (i+j)/2, (1+i+j)/2.$$

The base of this polyhedron is indicated schematically in Figure 1.

The groups stabilizing faces and vertices of the polyhedron are also shown, where a, b, d, e are as above and

$$c = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}, \quad f = \begin{pmatrix} -1 & j \\ j & 0 \end{pmatrix}$$

It is shown by Coxeter in [6] that vertices of Coxeter polyhedra in H^n are of simplicial type. For H^4 this means that there are four faces incident at each finite vertex. Each vertex group is the sense-preserving subgroup of the group generated by reflections in the faces of a tetrahedron in spherical space S^3. These elliptic tetrahedral orbifolds are given by their Coxeter diagrams in Thurston's manuscript [14]. In Table 1 the Coxeter diagrams, a presentation, and the order of each group is given. The orders were computed using John Cannon's computer language *Cayley* [5]. The generators are rotations along the edges of an elliptic tetrahedron, as shown in Figure 2.

Theorem 8. *The union of the presentations below is a presentation for* $PSL(2, \mathbf{Z}(i, j))$. *That is,* $PSL(2, \mathbf{Z}(i, j))$ *is a graph amalgamation product*

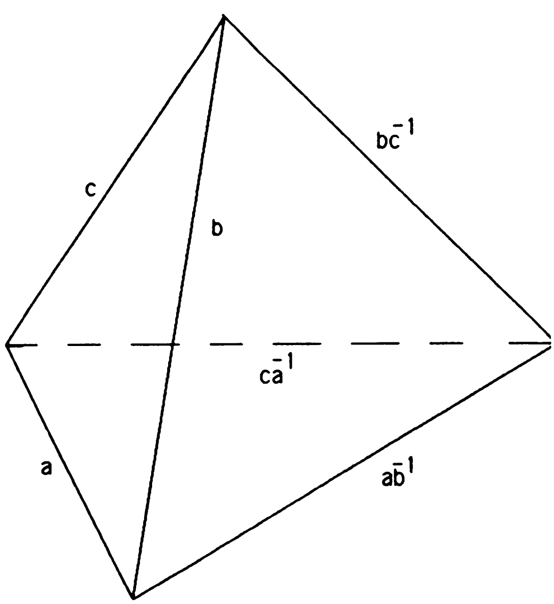


FIGURE 2

of the eight vertex groups.

$$\langle a, b, c\colon a^2 = b^2 = c^2 = (ab)^2 = (bc)^2 = (ca)^2 = 1\rangle\colon \textit{order } 8,$$
$$\langle b, c, d\colon b^2 = c^2 = d^3 = (bc)^2 = (cd)^2 = (db)^3 = 1\rangle\colon D_{12}, \textit{ order } 12,$$
$$\langle a, c, e\colon a^2 = c^2 = e^3 = (ac)^2 = (ce)^2 = (ea)^2 = 1\rangle\colon D_{12}, \textit{ order } 12,$$
$$\langle a, b, f\colon a^2 = b^2 = f^3 = (ab)^2 = (bf)^2 = (fa)^2 = 1\rangle\colon D_{12}, \textit{ order } 12,$$
$$\langle c, d, e\colon c^2 = d^3 = e^3 = (cd)^2 = (de^{-1})^2 = (ec)^2 = 1\rangle\colon S_4, \textit{ order } 24,$$
$$\langle b, d, f\colon b^2 = d^3 = f^3 = (bd)^2 = (df^{-1})^2 = (fb)^2 = 1\rangle\colon S_4, \textit{ order } 24,$$
$$\langle a, f, e\colon a^2 = f^3 = e^3 = (af)^2 = (fe^{-1})^2 = (ea)^2 = 1\rangle\colon S_4, \textit{ order } 24,$$
$$\langle d, e, f\colon d^3 = e^3 = f^3 = (de^{-1})^2 = (ef^{-1})^2 = (fd^{-1})^2 = 1\rangle\colon \textit{order } 96.$$

Proof. Note that if G is a finite volume group with one cusp and $\widetilde{G}$ is an extension of G such that the stabilizers G_∞ and $\widetilde{G}_\infty$ of the cusp are equal, then $G = \widetilde{G}$. This follows from the fact that a cusp must be represented on the boundary of a fundamental polyhedron.

The group G generated by a, b, c, d, e, f has the presentation above and is a subgroup of $PSL(2, \mathbf{Z}(i, j))$. This follows from Poincaré's theorem by the method described in the paper [4] once the dihedral angles are known. The classification of the groups follows from Coxeter and Moser [7]. The dihedral angles can be computed by considering the 3-dimensional case. For example, a hyperplane intersecting a sphere of radius 1 with distance $1/2$ from the center of the sphere to the hyperplane, intersects the sphere at an angle $\pi/3$.

The stabilizer G_∞ equals the stabilizer of ∞ in $PSL(2, \mathbf{Z}(i, j))$. The generators of G_∞ are the words of length two and having order two in the presentation above. These matrices also generate the stabilizer of ∞ in $PSL(2, \mathbf{Z}(i, j))$.

This completes the proof of the theorem.

Now as to a presentation for $PSL(2, \mathscr{H})$ where $\mathscr{H}$ is the *Hurwitz ring* of integral quaterions. The Hurwitz ring is generated over $\mathbf{Z}$ by $\{\zeta, i, j, ij\}$ where $\zeta = \frac{1}{2}(1 + i + j + ij)$ and $\mathscr{H}$ is a maximal order in the division ring of quaternions over Q. This ring satisfies a one-sided euclidean algorithm for division which $\mathbf{Z}(i, j)$ cannot (Vignéras [15, p. 91]) (see below). We have

$$\zeta = \tfrac{1}{2}(1 + i + j + ij), \quad \zeta' = \tfrac{1}{2}(1 - i - j + ij),$$
$$\zeta^* = \tfrac{1}{2}(1 + i + j - ij), \quad \overline{\zeta} = \tfrac{1}{2}(1 - i - j - ij),$$

with $\zeta\overline{\zeta} = \zeta'\zeta^* = |\zeta|^2 = 1$.

Let $h = \begin{pmatrix} \zeta ij & 0 \\ 0 & \zeta' ij \end{pmatrix}$. Then h takes $1, i, j$ to $j, 1, i$, respectively, and fixes i_3. So h has order 3 and maps the polyhedron for $PSL(2, \mathbf{Z}(i, j))$ onto itself. Furthermore, h conjugates a, b, c to b, c, a and d, e, f to e, f, d, respectively. This proves the next theorem.

Theorem 9. *$PSL(2, \mathscr{H})$ is a semidirect product of $PSL(2, \mathbf{Z}(i, j))$ and the cyclic group of order 3 generated by h.*

As a concluding remark we note that if n is sufficiently large then Vinberg [16] has shown that $PSL(2, \mathscr{C}_n(\mathbf{Z}))$, being arithmetic, is not of finite index in a reflection group.

7. The orbit of infinity

As for the modular and Picard groups $PSL(2, \mathbf{Z}(i, j))\infty$ is all of $\widehat{Q}^3$. In order to prove this we need several preliminary results.

Let $\mathscr{H}$ denote the Hurwitz ring generated over $\mathbf{Z}$ by $\{\zeta, i, j, ij\}$ where $\zeta = \frac{1}{2}(1 + i + j + ij)$. We term $\alpha, \beta \in \mathscr{H}$ *relatively prime* if $\alpha = \tilde{\alpha}d$ and $\beta = \tilde{\beta}d$ implies d is a unit in $\mathscr{H}$. This ring satisfies a one-sided euclidean algorithm.

Lemma 9 (Herstein [10, p. 331]). *If $\alpha, \beta \in \mathscr{H}$ then there exist $q, r \in \mathscr{H}$ with $\alpha = q\beta + r$ and $|r| < |\beta|$.*

Lemma 10. *If $\alpha, \beta \in \mathscr{H}$ are relatively prime then there exist $x, y \in \mathscr{H}$ such that $x\alpha + y\beta = 1$.*

Proof. Let $S = \{x\alpha + y\beta | x, y \in \mathscr{H}\}$ and consider an element d of minimal positive norm in S. By Lemma 9, there exist r, $\tilde{\alpha} \in \mathscr{H}$ such that $\alpha = \tilde{\alpha}d + r$ with $|r| < |d|$. This implies

$$\begin{aligned} r &= \alpha - \tilde{\alpha}d \\ &= \alpha - \tilde{\alpha}(x_1\alpha + y_1\beta) \quad \text{where } d = x_1\alpha + y_1\beta \\ &= (1 - \tilde{\alpha}x_1)\alpha + (-\tilde{\alpha}y_1)\beta. \end{aligned}$$

Now r belongs to S and this is a contradiction unless $r = 0$. The corresponding results for β proves the lemma.

Lemma 11. *If $\alpha, \beta \in \mathscr{H}$ are relatively prime and $\alpha^*\beta \in \mathbf{Q}^3$, then there exist $x, y \in \mathscr{H}$ such that $xy^* \in \mathbf{Q}^3$ and $x\alpha + y\beta = 1$.*

Proof. If $x_1\alpha + y_1\beta = 1$ then $(x_1 + \lambda\beta^*)\alpha + (y_1 - \lambda\alpha^*)\beta = 1$ since $\alpha^*\beta \in \mathbf{Q}^3$ iff $\beta^*\alpha = \alpha^*\beta$. We claim that if $\lambda = y_1x_1^*$ then $(x_1 + \lambda\beta^*)(y_1 - \lambda\alpha^*)^* \in \mathbf{Q}^3$. Observe that it suffices to show that

$$(x_1 + y_1x_1^*\beta^*)(y_1^* - \alpha x_1y_1^*) = (y_1 - y_1x_1^*\alpha^*)(x_1^* + \beta x_1y_1^*).$$

This is equivalent to

$$\begin{aligned} (1 - x_1\alpha)(x_1y_1^*) &+ (y_1x_1^*)(y_1\beta)^* - (y_1x_1^*)(\beta^*\alpha)(x_1y_1^*) \\ &= (y_1\beta)(x_1y_1^*) + (y_1x_1^*)(1 - x_1\alpha)^* - (y_1x_1^*)(\alpha^*\beta)(x_1y_1^*) \end{aligned}$$

and the result follows.

Theorem 10. $PSL(2, \mathbf{Z}(i, j))(\infty) = \widehat{Q}^3$ *and* $PSL(2, \mathscr{H})(\infty) = \widehat{Q}^3$.

Proof. The first statement follows from the second since $PSL(2, \mathbf{Z}(i, j))$ has index three in $PSL(2, \mathscr{H})$ with coset representatives $\{1, h, h^2\}$ and h fixes infinity. By Lemma 11, if $q \in \mathbf{Q}^3$ then there exists $T \in PSL(2, \mathscr{H})$ with $T\infty = q$ and the theorem is proved.

Unlike the situation in lower dimensions we have

Theorem 11. *If* $n \geq 3$ *then* $PSL(2, \mathscr{C}_n(\mathbf{Z}))(\infty) \neq \widehat{Q}^{n+1}$. *In particular there does not exist* $T \in (PSL(2, \mathscr{C}_n(\mathbf{Z})))$ *with* $T\infty = \frac{1}{2}(1 + i_1 + i_2 + i_3)$.

Proof. Suppose $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $T\infty = \frac{1}{2}(1 + i_1 + i_2 + i_3)$. If $d = 0$ then $|c| = 1$, which is impossible since $ac^{-1} = \frac{1}{2}(1 + i_1, + i_2 + i_3)$. Since $d \neq 0$

$$\begin{aligned} 1/|c|^2 &= |d|^2 \cdot |ac^{-1} - bd^{-1}|^2 \\ &= |d|^2[|ac^{-1}|^2 + |bd^{-1}|^2 - \mathrm{Re}(2ac^{-1})(\overline{bd^{-1}})] \\ &= |d|^2 + |b|^2 - \mathrm{Re}(2ac^{-1})(d\bar{b}) \in \mathbf{Z}. \end{aligned}$$

Thus $|c| = 1$: a contradiction.

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